VIII. APPENDIX: ENGLISH TRANSLATION OF CORNU'S ORIGINAL PAPER

Two Optical Methods for the Study of the Elasticity of Solid Bodies

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INTRODUCTION

The rigorous study of the elastic properties of solid bodies, especially those which, due to their rarity (such as natural crystals) are only obtained in small samples, presents extreme difficulties.

It is to the most delicate properties of light waves that I have addressed myself in an attempt to overcome them: to this end I have tried to establish two optical methods, apparently very different, but which, in essence, as we will see later, are theoretically equivalent and appear identical in precision.

The first, which I had initially believed to be much superior to the other, was described in 1869, in the *Comptes rendus de l'académie des sciences* (vol. LXIX p. 333): the second was only indicated by a note at the bottom of page 336 of the same communication which makes its principle known.

Both are based on the variation of optical phenomena produced by the deformation of a polished surface cut on the elastic body.

The flat surface of a parallelepiped blade is the shape that offers the easiest kind of observations.

The deformation modes used are:

1. The so-called circular bending: the blade is placed horizontally on two parallel supports and flexed by two equal and symmetrical weights: simple devices allow to perform the convex bending and the concave bending.
2. The torsion: the blade is placed on a support on one side and another on the other side. It is twisted in one direction or the other using symmetrical levers perpendicular to the torsion axis. The inverse and symmetrical bending and torsion modes have the advantage of eliminating several causes of error, and of doubling the precision.

The two optical methods used to determine the deformation of the originally flat surface verify EULER's theorem on the law of variation of the curvatures of the surface around a point. We immediately recognize the two main rectangular sections whose curvatures $\frac{1}{R}$ and $\frac{1}{R'}$ are almost always opposite sign: micrometric measurements then verify the relation.

$$\frac{\cos^2\omega}{R} + \frac{\sin^2\omega}{R'} = \frac{1}{\rho} \tag{1}$$

usable whenever one needs to measure the curvature $\frac{1}{\rho}$ of a normal section making the angle ω with the principal section of the curvature $\frac{1}{R}$.

1ST METHOD

Based on the use of Newton rings

We observe with monochromatic light the rings produced by the air blade located between the deformed elastic surface and the most often flat surface of a converging lens of 40 to 50 cm focal length carried on three screw points.

The optical phenomenon, observed almost normally with the yellow light of sodium (according to the FIZEAU device) is very striking by its geometric elegance. The flat surface deformed by bending or torsion being with opposite curvatures determines a system of hyperbolic rings having the same asymptotes: we thus intuitively verify on the one hand the rectangularity of the principal sections (parallel to the principal axes of the hyperbolas) and on the other hand the law of variation of the curvatures of EULER's theorem; the hyperbolas in fact realize the indicators of CH. DUPIN, to which the sections of the surface are reduced by planes parallel to a tangent plane at very small distances from the point of contact, because distances are equal to 0.5, or, 0.589 μ m or a third of micron.

This system of hyperbolas, at the same time, paints to the eyes the shape of the surface according to the mode of topographical representation, the scale of the heights of the horizontal sections being precisely equal to half a wavelength.[1]

The survey of these rings can be done at the focus of a small observation telescope equipped with a wire micrometer: but the precision of the points becomes much greater by using photography.

For this purpose, the telescope is replaced by a photographic apparatus aimed almost normally at the surface of the air blade where the rings are formed; the light source is an induction spark leaping between two magnesium poles. The spark is placed in a slightly oblique direction on the axis of the aforementioned lens so that its conjugate image reflected by the interference surfaces is made on the photographic objective.

In a few seconds we obtain a snapshot of the rings (negative snapshot) because the most intense radiation is sufficiently monochromatic; in reality it is an ultraviolet triplet whose average wavelength is $\lambda = 0.383\mu m$.

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[1] The lower surface of the lens can be chosen not only flat but slightly convex or concave: we then obtain all the forms of indicators described by DUPIN, ellipses, systems of parallel lines and hyperbolas.

These images, which can withstand high magnifications, can be measured at leisure using a microscope with a micrometer slide.

The precision of the points can go very far and reach at least $\frac{1}{20}$ of a ring, which corresponds to $\frac{1}{40} \times 0.383\mu m = 0.0095\mu m$ or one hundredth of a micron in the assessment of the relative deformations normally on the surface.[2]

The use of NEWTON rings has the advantage of providing an overview of the simultaneous deformations of all points of the elastic surface, independently of the bending of the supports, or of the support of the auxiliary lens used to produce the rings.

The actual size of the rings is obtained by means of a rectangular grid of a determined dimension, traced with a diamond or hydrofluoric acid on the flat surface of the auxiliary lens. The photographic image of this grid also makes it possible to correct the effect of the slight obliquity of the beams.

Micrometric measurements

We generally just measure the diameters of rings of the same order along the two main rectangular sections. As the radii of curvature are very large compared to the thickness e of the air blade corresponding to the middle of diameter c of the ring considered, we have for each main section the well-known relation:

$$c^2 = 8Re \quad \text{and} \quad c'^2 = 8R'e \tag{2}$$

If we give each ring an order number i starting with $i = 1$ for the first from the center, the diameter c_i will correspond to the thickness e_i: now e_i is equal to an integer i plus a fraction of half-wavelengths. Therefore:

$$c_i^2 = 8R(i + \phi)\frac{\lambda}{2} = 4R\lambda i + 4R\lambda\phi \tag{3}$$

expression of form:

$$c_i^2 = ai + b, \qquad \text{by posing} \begin{cases} 4R\lambda = a \\ 4R\lambda\phi = b \end{cases} \tag{4}$$

The observations give i and c_i; it is a question of deducing a and b. We are naturally to use the method of least squares, because the number of diameters e_i measured is most often quite large. Let n be this number; we will have the condition:

$$\sum_1^n (ai + b - c_i^2)^2 = \text{minimum}, \tag{5}$$

which is reduced by equalling to zero the coefficients of da and db obtained by differentiating the above equation, to two linear equations in a and b which determine these two parameters:

$$\begin{cases} a\sum i^2 + b\sum i - \sum ic_i^2 = 0 \\ a\sum i + bz - \sum c_i^2 = 0 \end{cases} \tag{6}$$

The sum being taken from the equations,

$$\begin{cases} \sum i = \frac{n(n+1)}{2} \\ \sum i^2 = \frac{n(n+1)(2n+1)}{6} \end{cases} \tag{7}$$

Solved with respect to a and b these equations give:

$$a = \frac{2\sum ic_i^2 - (n+1)\sum c_i^2}{\frac{(n-1)n(n+1)}{6}}, \quad b = \frac{(2n+1)\sum c_i^2 - 3\sum ic_i^2}{\frac{(n-1)n}{2}} \tag{8}$$

In the numerical application of these formulas we encounter some simplifications which make their use very simple.

Elimination of accidental errors of pointing

In the above equation 4 the diameters c_i enter by their squares so that an accidental error of δc_i is included by the product $2c_i\delta c_i$, which seems to exaggerate the influence of the error of increasing diameters.

A series of direct observations showed me that the average pointing error on the rings is substantially proportional to the width of the average interval of the consecutive rings; we therefore have the condition:

$$\delta c_i = k(c_{i+1} - c_i) = k\frac{c_{i+1}^2 - c_i^2}{c_{i+1} + c_i} = \frac{k}{2}\frac{c_{i+1}^2 - c_i^2}{2(c_{i+1} + c_i)} \tag{9}$$

hence:

$$\frac{1}{2}(c_{i+1} + c_i)\delta c_i = \frac{k}{2} \tag{10}$$

it is therefore by their squares that the measured diameters must enter into the condition equation to give all the observations the weight which suits them.

It is understood that each measurement of c_i is an average of several repeated measurements to mitigate random pointing errors: the calculation of coefficients a and b by the least squares method aims to eliminate their influence.

Systematic errors

The values c_i, before being treated as just explained, need to be corrected for a systematic error whose origin is as follows. When observing with a wire micrometer, one inevitably leads to pointing to the middle of the two edges of the ring (dark or light) instead of pointing to

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[2] With certain gelatin-bromide plates, when the induction spark is not suitable, the violet radiation $\lambda = 448$ superimposes a system of rings which alters the purity of the curves by giving them a periodic 6:7 appearance: this radiation is eliminated using a layer of collodion with $\frac{1}{10}$ of chrysoidine which is spread on the flattest surface of the photographic lens.

the position of the minimum or maximum of intensity, which is always uncertain; the asymmetry of each ring makes this exact assessment almost impossible, so it is necessary to calculate the position of this minimum or maximum from the pointing which corresponds to the middle of the visible edges of the ring. The geometric discussion of this condition leads to the following rule.

The square of each apparent diameter (defined by the four midpoints of the four edges of the ring) will be corrected by adding the square of the apparent width (mean distance of the contiguous edges) of the ring.

These are the corrected values c_i which we introduce into the above equations 6 and 8.

Calculation of the two principal curvatures

We operate as has just been said in the two principal directions, i.e. along the principal axes of the system of hyperbolas; we conclude the values $a = 4R\lambda$ along one of the axes and $a' = 4R'\lambda$ along the other; the values b and b' serve as verification, because we must have the relation $\phi + \phi' = 1$, the series of conjugate hyperbolas on either side of the asymptotes corresponding to the succession of thickness of the air blade which vary in a continuous manner according to the series of whole numbers.

We thus calculate the curvatures $\frac{1}{R}$ and $\frac{1}{R'}$, which enter into the formulas of the theory of elasticity.

If, as a result of any asymmetry, the principal axes do not coincide with the edges of the parallelepiped blade, the angle ω of deviation is measured, and EULER's theorem allows us to calculate the curvatures in the planes of symmetry of the blade, or in the sections designated by the theory; thus in the phenomenon of torsion the sections of interest are at $\alpha = +45$ from the axis of the blade, while in bending $\omega = 0$ or 90.

Substitution of a purely optical method for the micrometric survey of rings

The photographic images of the rings form real diffraction gratings capable of forming focal images when they are interposed on the path of a beam of parallel rays.[3] For this purpose a collimator is provided whose usual slit is replaced by a very small hole illuminated by a very bright light using a collective lens.

The interposed image transforms the parallel beam into two series of astigmatic beams forming linear foci parallel to the principal axes of the hyperbolas. The distance from the surface of the image of these focal images is precisely proportional to the radius of curvature of the corresponding principal section and inversely proportional to the wavelength of the illuminating light.

The multiplicity of focal images does not cause any ambiguity because these images are very easily distinguished from one another; first by their direction which is always perpendicular to the main section whose curvature is sought; then by their distance from the surface of the image which is proportional to the series of whole numbers:

$$-3, -2, -1, 0, +1, +2, +3, \ldots$$

the + sign corresponds to the convergent foci, the - sign to the divergent foci, i.e. located beyond the diffractive surface; moreover they are most often reduced to two, corresponding to -1 and +1 as a result of the erasure of the others.

I will not dwell on the use of this method, which would be wonderfully elegant and simple if it were not somewhat difficult to implement in practice. The difficulties arise from several causes:

1. The photographs must be obtained on mirrors with very parallel faces so that refraction does not disturb the movement of the diffracted beams and not on commercial photographic plates which are common panes.

2. The light emitted by the collimator must be both very intense and noticeably monochromatic.

3. The measurement of converging focal distances is very direct and very easy; but that of diverging foci (necessary to eliminate certain causes of error) requires complex optical devices whose precision is uncertain.

All these conditions, although achievable in a comfortably installed laboratory, are, ultimately, more complicated than simple micrometric observations which are made at leisure, without any additional manipulation.

It was, however, useful to point out this synthetic mode of measurements which, in certain cases, would significantly shorten the determination of the principal elements of a deformed elastic surface of which one has the *topographical representation.*

The second optical method that I want to describe fulfills the same goal in a more direct way but, on the other hand, it does not preserve, like the first, the geometric image of the surface studied.

All the details I have just set out are not only the result of theoretical research: they derive from a very large number of experimental determinations.

I must confess, however, that I have never been entirely satisfied, and this is what has prevented me from publishing the figures obtained in the numerous measurements made on slides of supposedly isotropic substances, common glasses, crown and flint glass, steel, copper and crystallized substances (fluorite, rock salt, alum and quartz).

[3] Proceedings of the Academic of Sciences. Volume LXXX p. 615. 1875. French Association for the Advancement of Science, Nantes Congress p. 376.

Apart from the difficulty inherent in obtaining suitable materials, we find ourselves placed between two pitfalls: if the relative transverse dimensions of the blades are very small as required by the formulas of the theory of elasticity, the deformed surfaces are so narrow that the number of usable rings is too small, at least in the transverse direction, to obtain sufficient precision. If, in order to increase the field of visible rings, we widen the blades in relation to their thickness, we risk no longer being in the simple conditions in which the equations of elasticity have been integrated. This can be recognized in a somewhat crude way, it is true, but very suggestive, by bending or twisting blades of various widths cut from a rubber plate 10 to 15 millimeters thick: the outer faces take on shapes incompatible with theoretical predictions.

However, I do not consider these difficulties to be insurmountable: [4]I have tried on several occasions to improve my first attempts, but unfortunately I lacked time and, above all, assistance; it is impossible to successfully complete such meticulous work alone[5].

I therefore had to be content to discuss the experimental methods in order to prepare the way for observers who would have the necessary resources, in terms of equipment and personnel, to implement them.

2ND METHOD: BASED ON THE USE OF FOCAL IMAGES BY REFLECTION

The method based on the observation of NEWTON's rings had seemed to me, at first, to be so marvelously precise (due to the smallness of the ultraviolet wavelengths used[6]) that I had not hesitated to attribute to it a priori an incontestable superiority.

So I remained until 1890 without seeking to develop the method of focal images produced by reflection on the deformed flat surface of elastic blades indicated in my work of 1869.

It was following studies of a completely different kind that I clearly perceived the precision of this mode of exploration which is in no way inferior, at least theoretically, to the observation of Newtonian rings and which has the advantage of being more direct. I therefore studied a device which allows measurements similar to those of the first method to be made routinely.

We operate as before on a horizontal parallelepiped blade whose upper face is flat and polished, by bending or by twisting; above this blade and at a very small distance rests on three screw points an achromatic objective with a focal length of 25 to 30 cm. .

The mode of observation is comparable to that which astronomers use to observe the nadir on the flat surface of a mercury bath, or that which physicists sometimes call *autocollimation*.

A luminous point placed at the principal focus of the objective gives beyond a beam of parallel rays; this beam is normally reflected on the flat surface of the plate not yet deformed, is refracted again through the objective, and will form in the principal focal plane a luminous point, image of the source and placed next to it. If we now come by bending or twisting to deform the surface of the elastic plate, the reflected beam becomes astigmatic; the single point focus is transformed into two rectangular linear foci, but separate, in accordance with MALUS's theorem.

It is from the position and orientation of these two focal images relative to their position where they form a single point image that we deduce both the value of the principal curvatures of the deformed surface and the direction of the principal sections which correspond to them.

To understand how these two types of measurement can be carried out, it is necessary to briefly describe the experimental device which provides the light source and which allows its two reflected images to be observed.

As a source it is very convenient to choose a luminous line (edit needed: symbol missing?) between dark lines formed by a $\frac{2}{100}$ millimeter glass wire illuminated by transparency using a collecting lens returning light towards the objective in the direction of the main axis of this objective. The wire is stretched in the center of a hollow alidade movable on a divided circle. Behind this wire is placed in the direction of the main axis of the objective a microscope sliding longitudinally on a carriage with the aid of a rack and pinion[7]. When the apparatus is suitably adjusted, the reflected image of the luminous wire is painted next to the material wire with great clarity when the direction of this wire is parallel to one of the main sections of the deformed slide. If this condition is

[4] Thus the number of rings can be increased by replacing the plane surface of the converging lens with a convex or concave surface of known curvature. The rings become circular, elliptical, rectilinear, following all the varieties of indicators and can give in better conditions the variations of curvature of the deformed surface.

[5] I must, however, mention the zeal and skill with which a young Russian physicist, Mr. WOULF, since professor of mineralogy in Warsaw, was kind enough to help me work on this method during the few months spent in my laboratory at the Polytechnic School: he carried out in particular with great care not only the revision of the calculation procedures, but also the somewhat thankless task of verifying the accuracy of the method by determining by means of rings the value of the curvature of several convex and concave surfaces determined directly either by the focal images or by the use of my reflection lever (Journal de Physique 1er Série tome IV. p. 7).

May I be permitted to express to him here my very sincere thanks.

[6] With a quartz lens and a quartz-fluorine photographic objective we can use the quadruple ultraviolet line $\lambda = 280$ which gives even tighter rings and therefore significantly increase the delicacy of the method: we would certainly go even further by taking certain precautions in the choice of the actinic source.

[7] Proceedings of the Academic of Sciences. Volume LXXX p. 615. 1875. French Association for the Advancement of Science, Nantes Congress p. 376.

not fulfilled, the sharpness remains defective despite the variation of focus of the microscope, but by rotating the alidade in one direction or the other[8] we find an orientation of the wire for which the reflected image presents an admirable sharpness; it is the direction of one of the two main sections. The other section is obtained immediately by rotating the alidade by a right angle. We thus verify the first part of EULER's theorem.

As for the determination of the curvatures, it is obtained using the following operations:

1. The principal focal length f and the position of the nodal points of the lens are carefully determined with an appropriate footometer (see Journal de Physique 1st Series. Volume VI, p. 276 and 308).

2. The objective being placed on its three points, the elastic blade to be studied is replaced by a perfectly flat glass; the position of the principal focal plane of the objective is then determined by the condition that the wire and its image are at the same point in the focal plane of the microscope. This is an adjustment that is made once and for all and of which the trace is kept by reading on the carriage which carries the microscope the graduation corresponding to this position.

If now we substitute a deformed elastic blade for the flat glass, the image of the wire held in the previous position will be in another focal plane: the difference in position of this image will be measured by the displacement of the microscope focused on one axis or the other of the aforementioned linear images. We determine, by a double trial and error, the two rectangular azimuths of the wire which give the sharpest images and we read on the graduation of the carriage the corresponding positions of the microscope.

The surface of the elastic blade is almost in contact with the outer surface of the lens, we can almost always consider it as coinciding with the outer nodal point. We will nevertheless begin by assuming that the distance of this surface is finite and equal to d. Let p be the distance from the luminous wire to the inner nodal wire, p' that of conjugate focus with respect to the lens counted from the outer nodal point and in the same direction:

We will have

$$\frac{1}{p} - \frac{1}{p'} = \frac{1}{f}. \tag{11}$$

Let s and s' be the distances of the conjugate images of the reflecting surface of radius R assumed to coincide with the exterior nodal point and counted in the same direction; we will have

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R}. \tag{12}$$

Finally q and q' the distances analogous to p and p' for the return of the beams reflected through the lens, we will have in the same way

$$\frac{1}{q} - \frac{1}{q'} = \frac{1}{f}. \tag{13}$$

The conditions for binding the conjugate images are:

$$s = p' + d \quad , \quad s' = q' + d. \tag{14}$$

The elimination of s and s' is easily done and we find

$$\frac{2}{R} = \frac{f-p}{d(f-p)+fp} + \frac{f-q}{d(f-q)+fq'} \tag{15}$$

equation which provides the curvature of the blade in the chosen section.

But most often we find that the products $d(f-p)$, $d(f-q)$ are negligible compared to the terms fp and fq, because d is very small and on the other hand p and q are close to f; it follows that we can assume $d = 0$ and then the equation is reduced to

$$\frac{1}{p} + \frac{1}{q} = 2\left[\frac{1}{f} + \frac{1}{R}\right] = \frac{2}{\rho}, \tag{16}$$

which we would obtain directly by adding member to member 11, 12 and 13 by making $d = 0$ in 14.

From which we conclude this very simple result.

Under the above conditions of the optical device, the system formed by the objective and the reflecting surface along a main section of radius of curvature R is equivalent to a coaxial mirror in coincidence with the interior nodal point whose curvature would be

$$\frac{1}{\rho} = \frac{1}{f} + \frac{1}{R}. \tag{17}$$

Let us imagine that we modify the deformation of the blade without changing the position p of the wire, the light source; its conjugate image will pass to the distance q, as a result of the change in the radius of curvature R becoming R_1; we will therefore have the condition

$$\frac{1}{p} + \frac{1}{q_1} = 2\left(\frac{1}{f} + \frac{1}{R_1}\right). \tag{18}$$

Removing member by member to eliminate p which at the same time eliminates f, it comes

[8] We are guided in this test by some asymmetrical part of the image which appears blurred in the desired direction so that the groping in direction is so to speak zero: we must naturally vary the focus at the same time to obtain the maximum sharpness.

$$\frac{1}{q} - \frac{1}{q_1} = 2\left(\frac{1}{R} - \frac{1}{R_1}\right). \tag{19}$$

Fundamental equation of the method, because it determines the variations of curvature which enter into the formula of the theory of elasticity.

Observation gives q and q_1 directly by reading the scale along which the microscope carriage moves when pointing:

- the principal focal plane such that $p = q$
- the focal plane q_1 of the linear image after deformation.

The same equation applies to any two or more deformations, under the influence of two or more successive and unequal bending or torsion moments; for each principal section we have for two successive deformations a condition of the form

$$\frac{1}{q_0} - \frac{1}{q_1} = 2\left(\frac{1}{R_0} - \frac{1}{R_1}\right), \quad \frac{1}{q_0'} - \frac{1}{q_1'} = 2\left(\frac{1}{R_0} - \frac{1}{R_1}\right) \tag{20}$$

independent of the value of the main focal length f of the lens.

In the law of bending of isotropic blades the ratio of these variations in curvature gives precisely the POISSON coefficient σ (ratio of transverse contraction to longitudinal expansion[9]).

In the case of isotropic blade torsion, the main sections are at 45 to the blade axis and the curvature variations are equal and of opposite sign (equilateral hyperbolas as indicator, focal image deflections equal and of opposite sign). The magnitude of the focal image deflection allows the calculation of the coefficient μ of LAMÉ theory[10].

There may be slight asymmetries in the orientation of the main sections due to a defect in the size or unevenness in the supports.

The divided circle on which the alidade which carries the glass wire moves gives the means of measuring the angle ω which with the help of EULER's equation provides the value of the curvatures in the principal sections: there is no need to insist on the somewhat minute corrections which practically do not present any difficulty.

This method of focal images has served me especially to study the value of the POISSON coefficient on glass slides and rods. This coefficient, according to my determinations, has always been close to $\frac{1}{4}$, but most often slightly lower than this fraction.

I would have liked to elucidate the cause of this divergence which I persist in believing, until further notice, to be accidental: I attribute it either to the heterogeneity of the vitreous materials used, or to the difficulty already noted above of fulfilling the conditions of exiguity of the transverse dimensions required by the theory. The heterogeneity of the glass plate is indisputable; polarized light highlights it; on the other hand, sawing or polishing introduces into the surface layers a forced molecular state that interference observations have allowed me to confirm with certainty.

The divergence reported does not depend on the mode of observation; because, thanks to the help of Mr. Woulf I was able to show that the same rod offers the same value of the coefficient σ by the method of rings as well as by that of focal images.

Although I have not had the opportunity to make precise measurements by this second method on crystallized slides, I have noted without difficulty that the smallness of the dimensions of the samples in the form of slides accommodates both the observation of focal images and those of Newtonian rings so that from the point of view of experimental facilities the two methods are equivalent. We will see that as regards the precision of the results the equivalence continues theoretically with a very high probability.

EQUIVALENCE OF THE TWO METHODS IN TERMS OF MEASUREMENT PRECISION

Despite the apparently essential difference between the two devices, it can be shown that their chances of precision are substantially equivalent when they are used under the same geometric conditions. The two methods are in fact intended to measure the deflection e of a circular arc of radius R corresponding to a chord c; between these three elements exists the relation 2 which we put in the form

$$e = \frac{c^2}{8R}. \tag{21}$$

We will demonstrate that the error made on this arrow is exactly of the same order with the two measurement modes.

With NEWTON rings $e = n\frac{\lambda}{2}$, corresponding to the diameter c_n of the n^{th} ring, the absolute error made on n is between $\frac{1}{10}$ and $\frac{1}{20}$ the width of a ring; let us adopt a more favorable

$$\delta e = \frac{\lambda}{2}\delta n. \tag{22}$$

If we admit that $\lambda = 0.38\mu m$ (magnesium spark)

$$\delta = 0.19\mu m \times \frac{1}{20} = 0.0095\mu m \tag{23}$$

or 0.01 micron.

[9] St. VENANT. Torsion of prisms, (volume IV, Foreign scholars).
[10] LAMÉ. Lessons on Elasticity and St. VENANT, Torsion of prisms.

With the method of focal images e is easily calculated from the $q_1 - q_0$ offset produced by the substitution of the surface of radius R for the flat glass observed with the objective of focal length f.

We have according to formula 20, by making $q_0 = f$, $R_0 = \infty$ and $R = R_1$,

$$\frac{1}{q_0} - \frac{1}{q_1} = -\frac{2}{R} \quad \text{or } q_1 - q_0 = 2\frac{q_0 q_1}{R}. \tag{24}$$

Or $q_0 = f$ and q_1 differ little; we can therefore for the approximate evaluation of the error replace $q_0 q_1$ by f^2. On the other hand, let m be the ratio of the focal length f of the lens to its free aperture c:

$$f = mc, \quad \text{hence } q_0 q_1 = f^2 = m^2 c^2 \tag{25}$$

We conclude

$$q_1 - q_0 = -2\frac{m^2 c^2}{R} \quad \text{and } e = \frac{c^2}{8R} = \frac{q_0 - q_1}{16m^2} \tag{26}$$

We obtain in good conditions the focus at $\frac{1}{20}$ of a millimeter; let $\delta(q_1 - q_0) = 0.05$mm. With m close to 12(the foot for sanding *(meaning unclear)*, following the rule of opticians) following the rule of opticians), therefore:

$$16m^2 = 16 \times 144 = 2304 \tag{27}$$

consequently

$$\delta e = \frac{\delta(q_1 - q_0)}{2304} = -\frac{1\text{mm}}{20 \times 2304} = \frac{1\mu m}{46.08} = 0.022\mu m. \tag{28}$$

or $\frac{2}{100}$ micron

So the approximation is of exactly the same order in both methods; it can reach the absolute value of $\frac{1}{100}$ microns under the most favorable conditions, conditions which are otherwise difficult to achieve.

In fact, it is necessary to use only optical devices of complete perfection; flat surfaces, lenses and objectives without aberrations, correct adjustments, etc. The addition of a large 45 mirror to bring the observation into a horizontal direction, much more convenient for the experimenter, risks causing astigmatism problems if the surface is not strictly flat.

From the point of view of the theory of light waves, the equivalence demonstrated numerically above is far from being unexpected. It is, in fact, the same wave that is observed in two different regions of its path; the mechanical and geometrical constitution of its movement is defined by equivalent elements in the whole space where it propagates; it would therefore not be difficult to show by reasoning analogous to that inaugurated by *FRESNEL* that any variation in the conditions of the initial vibratory movement on the surface where rings can be produced leads to a corresponding variation in the position of the real or virtual center of the wave which derives from it.

I will not stop at this demonstration; the indication of its theoretical value seemed to me to add further interest to these two methods and to recommend them to the attention of experimenters.